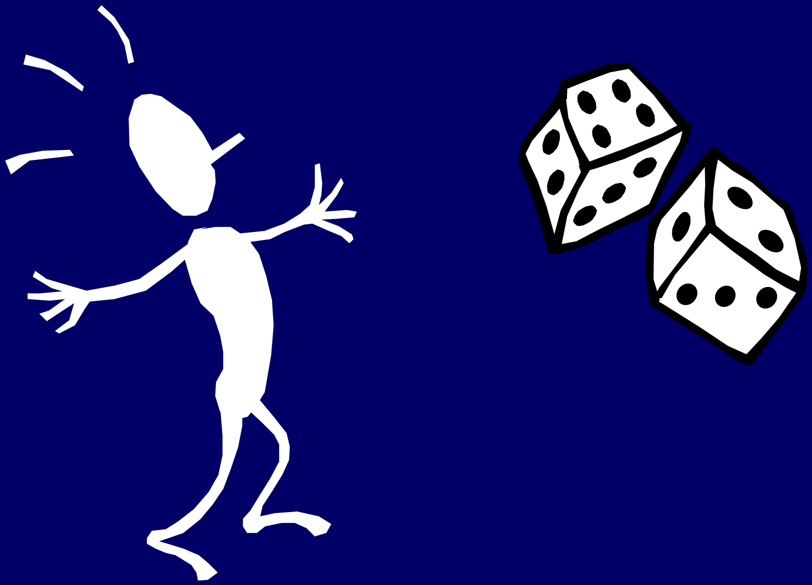
Randomness and Computation: Some Prime Examples



Great Theoretical Ideas In Computer Science

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Checking Our Work

Suppose we want to check p(x) q(x) = r(x), where p, q and r are three polynomials.

$$(x-1)(x^3+x^2+x+1) = x^4-1$$

If the polynomials are long, this requires n^2 mults by elementary school algorithms

-- or can do faster with fancy techniques like the Fast Fourier transform.

Can we check if p(x) q(x) = r(x) more efficiently?

Great Idea: Evaluating on Random Inputs

Let $f(x) = p(x)\, q(x) - r(x)$. Is f zero?

Idea: Evaluate f on a *random* input z.

If we get $f(z) = 0$, this is evidence that f is zero everywhere.

If f(x) is a degree 2n polynomial, it can only have 2n roots. We're unlikely to guess one of these by chance!

Equality checking by random evaluation

1. Fix a sample space $S=\{z_1, z_2,\dots, z_m\}$ with arbitrary points z_i, for $m=2n/d$.

2. Select random z from S with probability 1/m.

3. Evaluate $f(z) = p(z)\,q(z) - r(z)$

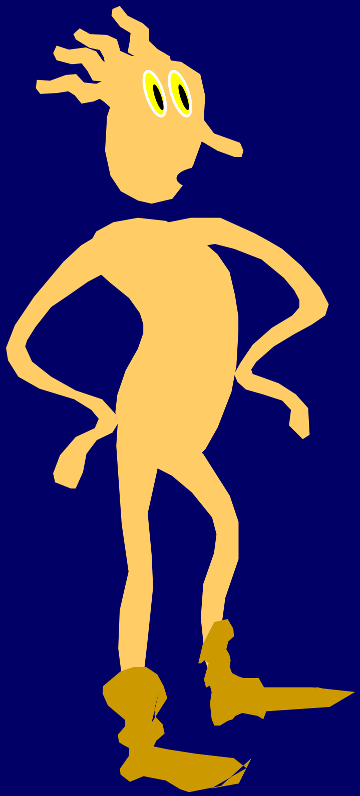
4. If $f(z) = 0$, output "equal" otherwise output "not equal"

Equality checking by random evaluation

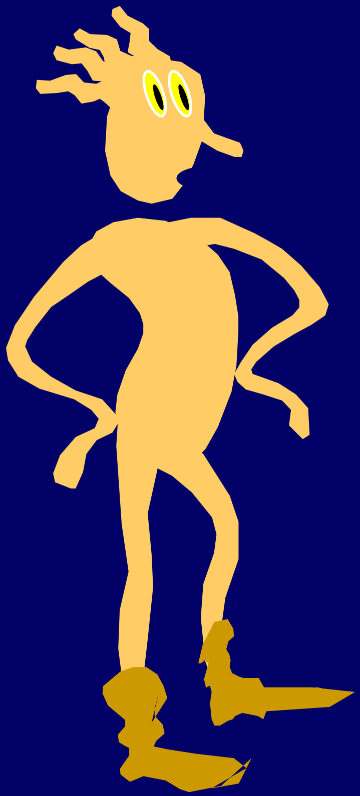
What is the probability the algorithm outputs "not equal" when in fact $f = 0$?

Zero!

If p(x)q(x) = r(x) , always correct!



Equality checking by random evaluation



What is the probability the algorithm outputs "equal" when in fact $f \neq 0$?

Let $A = \{z \mid z \text{ is a root of } f\}$.

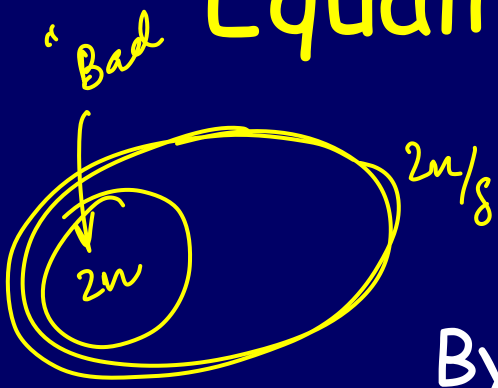
Recall that $|A| \leq$ degree of $f \leq 2n$.

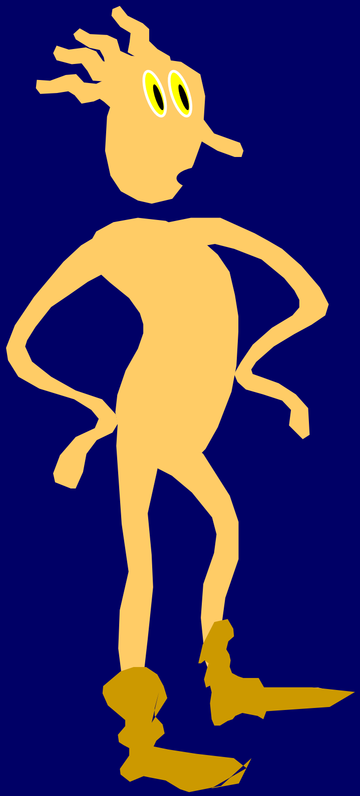
Therefore: $P(A) \leq 2n/m = d$.

size of set from which z is chosen

We can choose d to be small.

Equality checking by random evaluation





By repeating this procedure k times, we are “fooled” by the event

$$f(z_1) = f(z_2) = \ldots = f(z_k) = 0$$

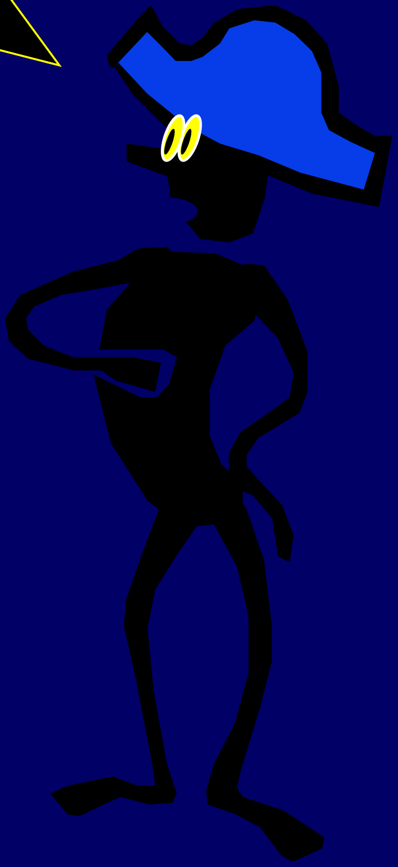
when actually $f(x) \neq 0$

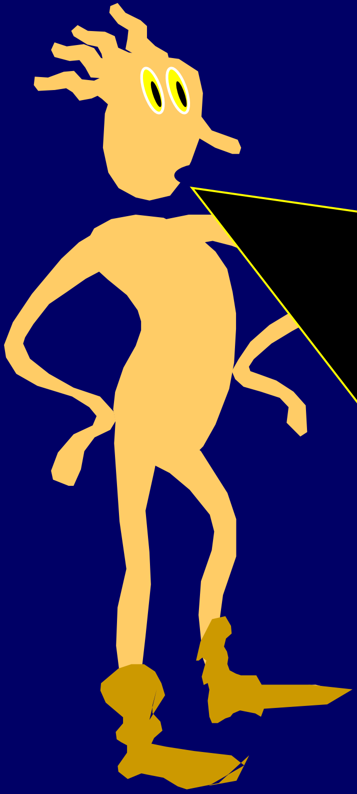
with probability no bigger than

$$P(A) \leq (2n/m)^k = d^k$$

$\left(\frac{1}{2}\right)^{100}$

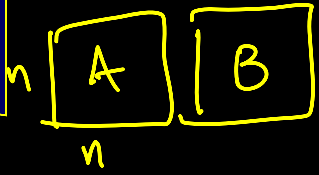
Wow! That idea could be used for testing equality of lots of different types of “functions”!





Yes! E.g., a matrix is just a special kind of function.

Suppose we do a matrix multiplication of two **nxn** matrices:



$$\underbrace{AB} = C$$

The idea of random evaluation can be used to efficiently check the calculation.

What does “evaluate” mean?

Just evaluate the “function” C on a random bit vector r by taking the matrix-vector product $C \times r$

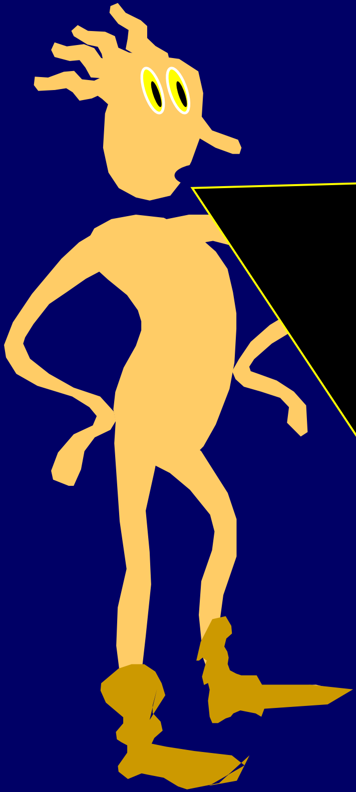
$AB = C$

random bit vector r

$ABr = Cr$? ✓

$$\begin{pmatrix} 1 & 0 & 3 & -4 & 8 \\ 7 & 0 & 0 & 2 & 9 \\ 13 & 5 & -6 & 0 & -7 \\ 1 & 6 & 21 & 9 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 9 \\ 7 \\ 31 \end{pmatrix}$$

$(AB-C)r \overset{?}{=} 0$



So to test if $AB = C$ we compute

$x = Br$, $y = \underbrace{Ax}$ (= Abr), and $z = Cr$

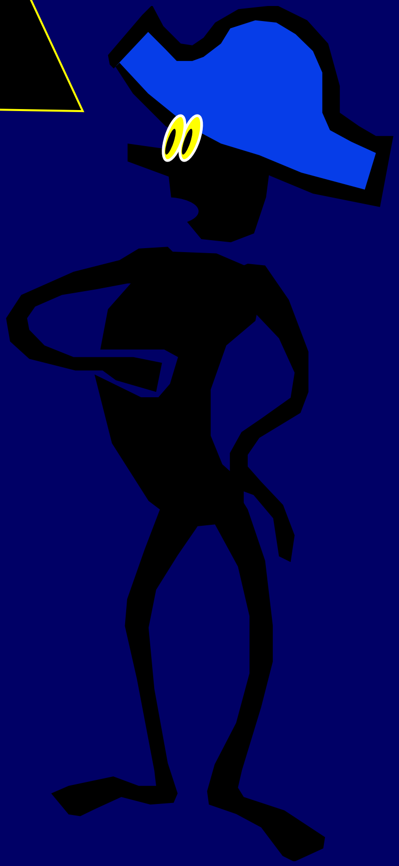
If $y = z$, we take this as evidence that the calculation was correct.

The amount of work is only $O(n^2)$.

Claim: If $AB \neq C$ and r is a random n-bit vector, then $\Pr(ABr = Cr) \leq \frac{1}{2}$.

Claim: If $AB \neq C$ and r is a random n-bit vector, then $\Pr(ABr = Cr) \leq \frac{1}{2}$.

So, if a complicated, fancy algorithm is used to compute AB in time $O(n^{2.236})$, it can be efficiently checked with only $O(n^2)$ extra work, using randomness!



"Random Fingerprinting"

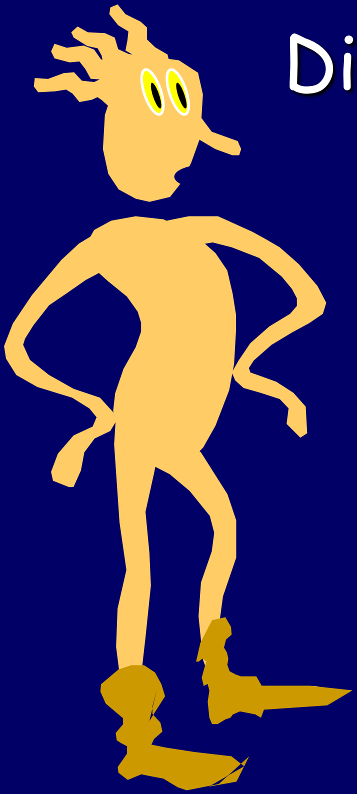
[Karp- Rabin]

Find a small random "fingerprint" of a large object.

- the value f(z) of a polynomial at a point z
- the value Cr at a random bit vector r

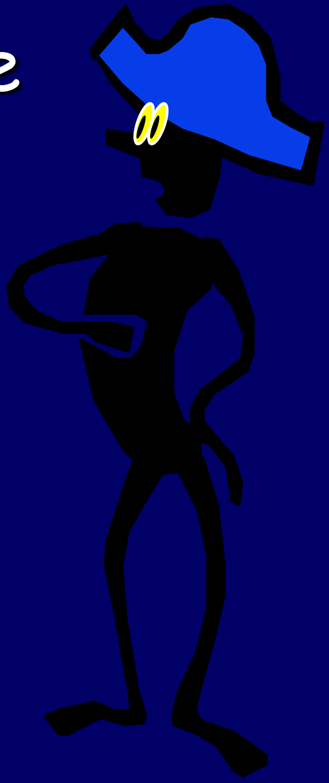
This fingerprint captures the essential information about the larger object: if two large objects are different, their fingerprints usually are different!

Earth has huge file X that she transferred to Moon. Moon gets Y.



Did you get that file ok? Was the transmission accurate?

Uh, yeah.



Earth: X

Moon: Y

Legendre

Gauss

Let π(n) be the number of primes between 1 and n. I wonder how fast π(n) grows?

Conjecture [1790s]:

$$\lim_{n\to\infty} \frac{\pi(n)}{n/\ln n} = 1$$

Their estimates

x	pi(x)	Gauss' Li	Legendre	x/(log x - 1)
1000	168	178	172	169
10000	1229	1246	1231	1218
100000	9592	9630	9588	9512
1000000	78498	78628	78534	78030
10000000	664579	664918	665138	661459
100000000	5761455	5762209	5769341	5740304
1000000000	50847534	50849235	50917519	50701542
10000000000	455052511	455055614	455743004	454011971

De la Vallée Poussin

J-S Hadamard

Two independent proofs of the Prime Density Theorem [1896]:

$$\lim_{n\to\infty}\frac{\pi(n)}{n/\ln n}=1$$

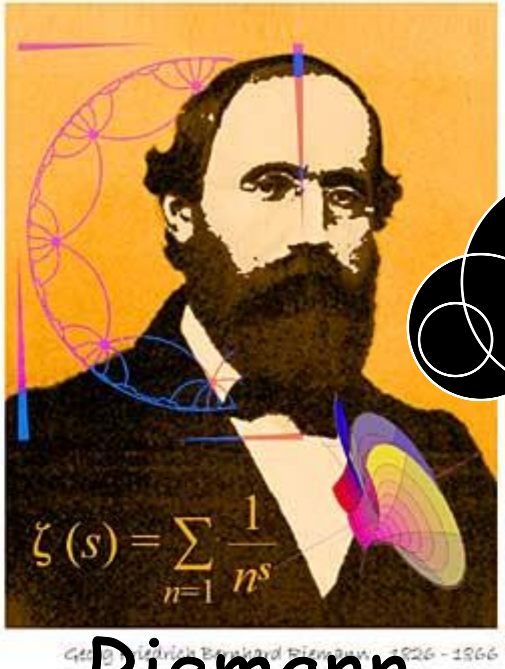
The Prime Density Theorem

This theorem remains one of the celebrated achievements of number theory.

In fact, an even sharper conjecture remains one of the great open problems of mathematics!

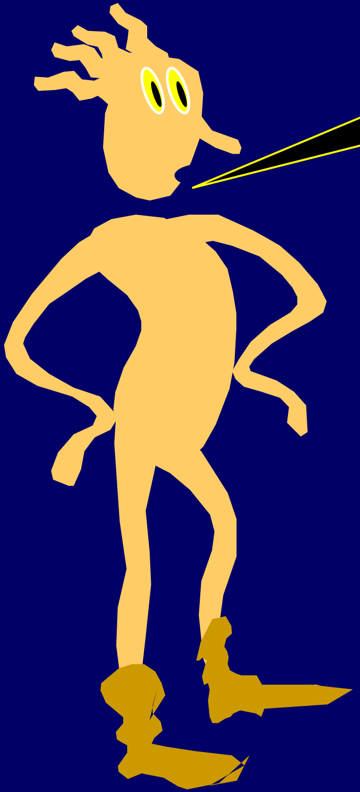
The Riemann Hypothesis [1859]

$$\lim_{n\to\infty}\frac{\pi(n)-n/\ln n}{\sqrt{n}}=0$$


$\zeta(s)=\sum_{n=1}\frac{1}{n^s}$
Georg Friedrich Bernhard Riemann 1826 - 1866

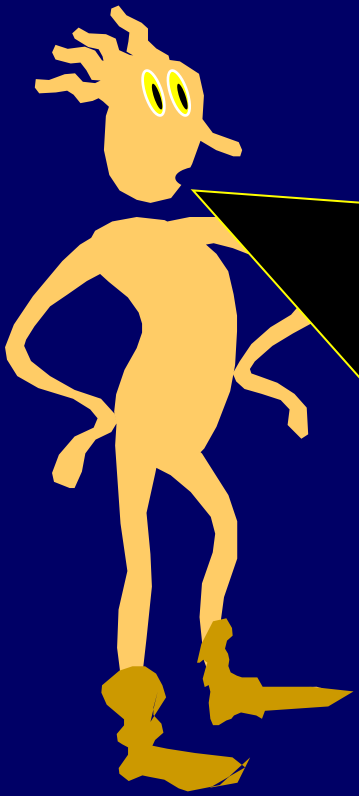
Riemann

Slightly easier to show
$\pi(n)/n \geq 1/(2 \log n)$.



# of primes between 1 & n

is at least $\frac{n}{2 \log n}$.

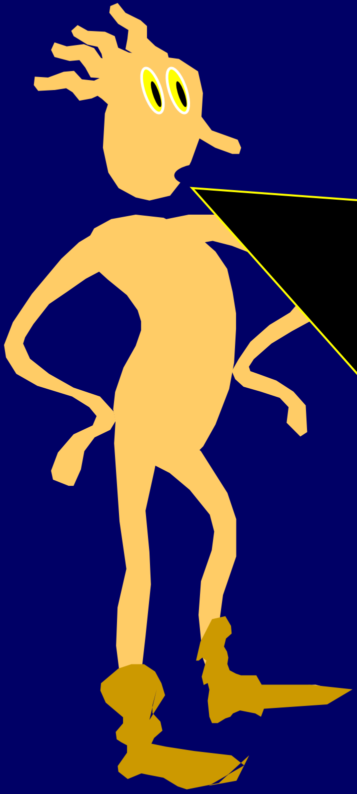


Random logn bit number is a random number from 1..n

$\pi(n) / n \geq 1/2\log n$

means that a random logn-bit number has at least a 1/2logn chance of being prime.

Random 1000 bit number is prime with prob $\geq \frac{1}{2000}$



Random k bit number is a random number from 1..2^k

$\pi(2^k) / 2^k \geq 1/2k$

means that a random k-bit number has at least a 1/2k chance of being prime.

Really useful fact

A random k-bit number has at least a 1/2k chance of being prime.

So if we pick 2k random k-bit numbers the expected number of primes on the list is at least 1

Picking A Random Prime

Many modern cryptosystems (e.g., RSA) include the instructions:

“Pick a random n-bit prime.“

How can this be done efficiently?

Picking A Random Prime

“Pick a random n-bit prime.”

Strategy:

1) Generate random n-bit numbers
2) Test each one for primality

Picking A Random Prime

"Pick a random n-bit prime."

1)Generate kn random n-bit numbers

Each trial has a ≥ 1/2n chance of being prime.

$n = 100$

$k = 10$

Pr[all kn trials yield composites]

$$\leq (1-1/2n)^{kn} = (1-1/2n)^{2n * k/2} \leq 1/e^{k/2}$$

Picking A Random Prime

"Pick a random n-bit prime."

Strategy:

1) Generate random n-bit numbers
2) Test each one for primality

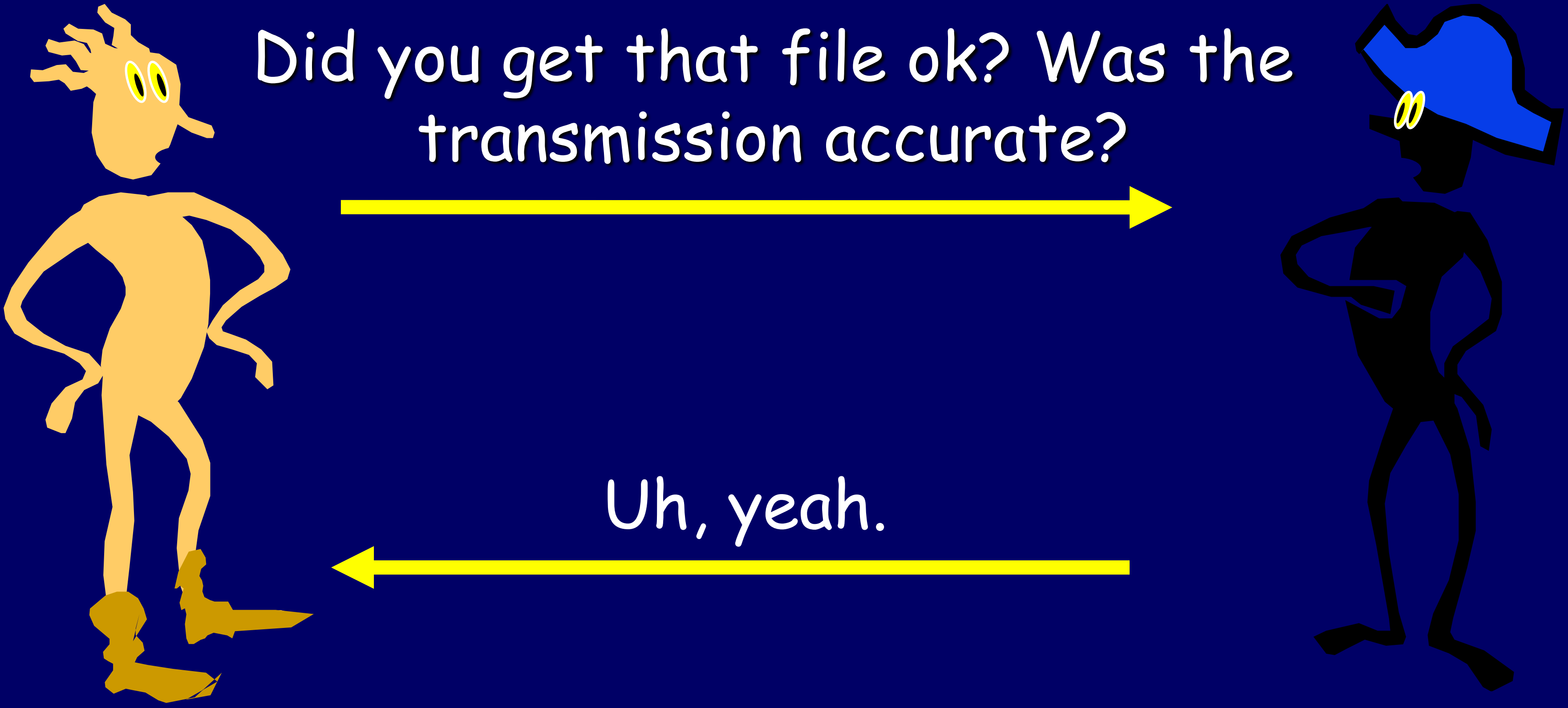
For 1000-bit primes, if we try out 10000 random 1000-bit numbers, chance of failing ≤ e^{-5}

Moral of the story

Picking a random prime is
"almost as easy as"
picking a random number.

(Provided we can check for primality.
More on this later.)

Earth has huge file X that she transferred to Moon. Moon gets Y.



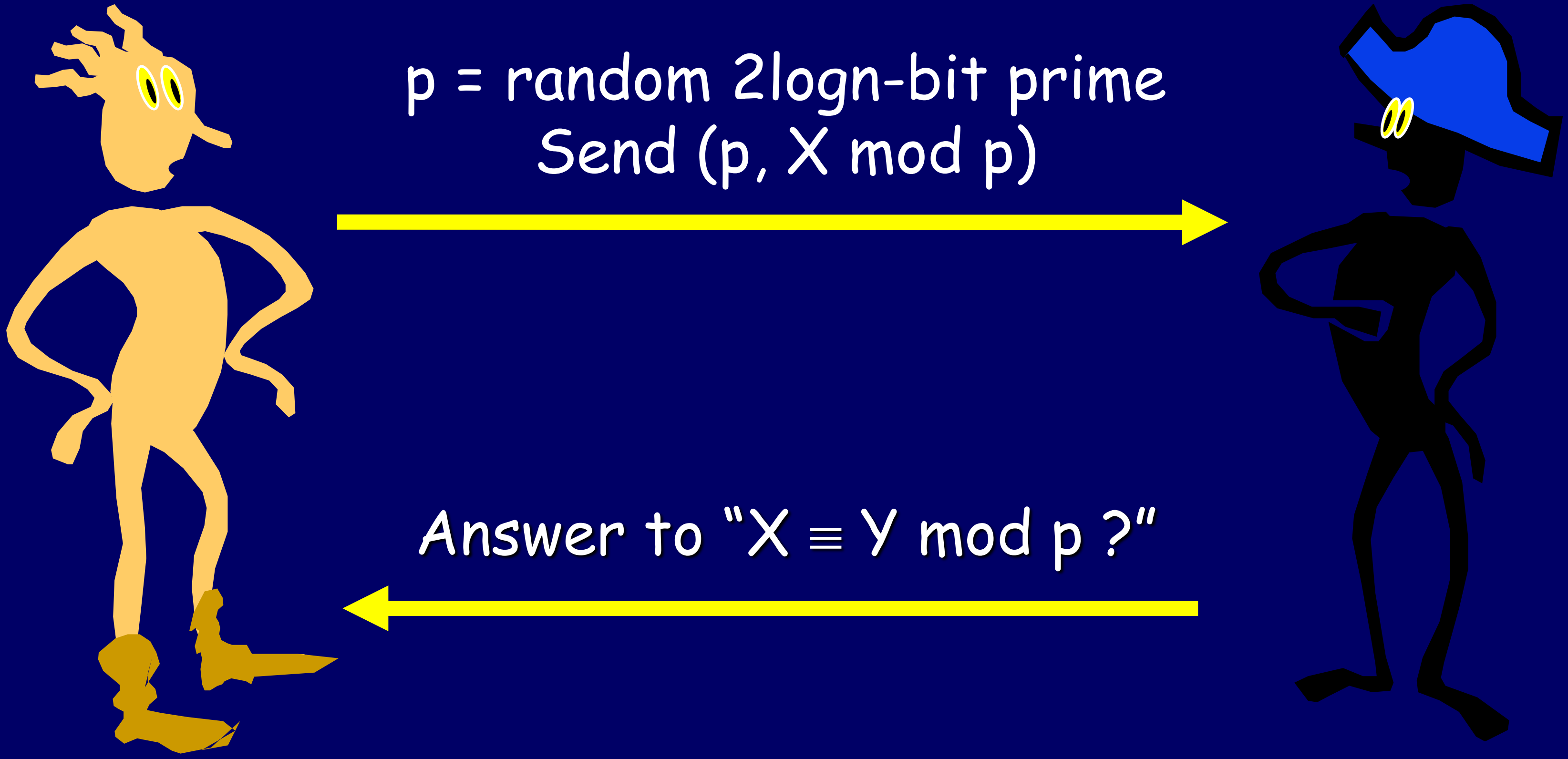
Did you get that file ok? Was the transmission accurate?

Uh, yeah.

Earth: X

Moon: Y

Are X and Y the same n-bit numbers?



p = random 2logn-bit prime
Send (p, X mod p)

Answer to “$X \equiv Y \bmod p$?”

Earth: X

Moon: Y

Why is this any good?

Easy case:

If $X = Y$, then $X \equiv Y \pmod{p}$

Why is this any good?

Harder case:

What if X $\cancel{\neq}$ Y? We mess up if p | (X-Y).

$$\boxed{\begin{array}{l} X \bmod p \\ = Y \bmod p \end{array}}$$

$\iff p \mid (x-y)$

Define Z = (X-Y). To mess up, p must divide Z.

Z is an n-bit number.

⇒ Z is at most 2^n. (duh.)

But each prime ≥ 2.

$Z = p_1 p_2 \cdots p_t$

Hence Z has at most n prime divisors. $\geq \underbrace{2 \cdot 2 \cdot \;\; 2}_{t}$

Almost there…

Z has at most n prime divisors.

How many 2logn-bit primes?

A random k-bit number has at least a 1/2k chance of being prime.

$\Rightarrow$ at least $2^{2\log n}/(2*2\log n) = n^2/(4\log n) \gg 2n$ primes.

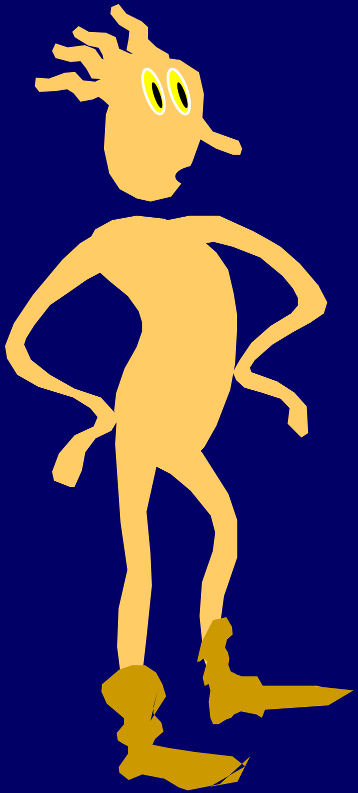
Only (at most) half of them divide Z.

$\Rightarrow$ make mistake with prob $\leq \frac{1}{2}$.

Theorem: Let X and Y be distinct n-bit numbers. Let p be a random 2logn-bit prime.

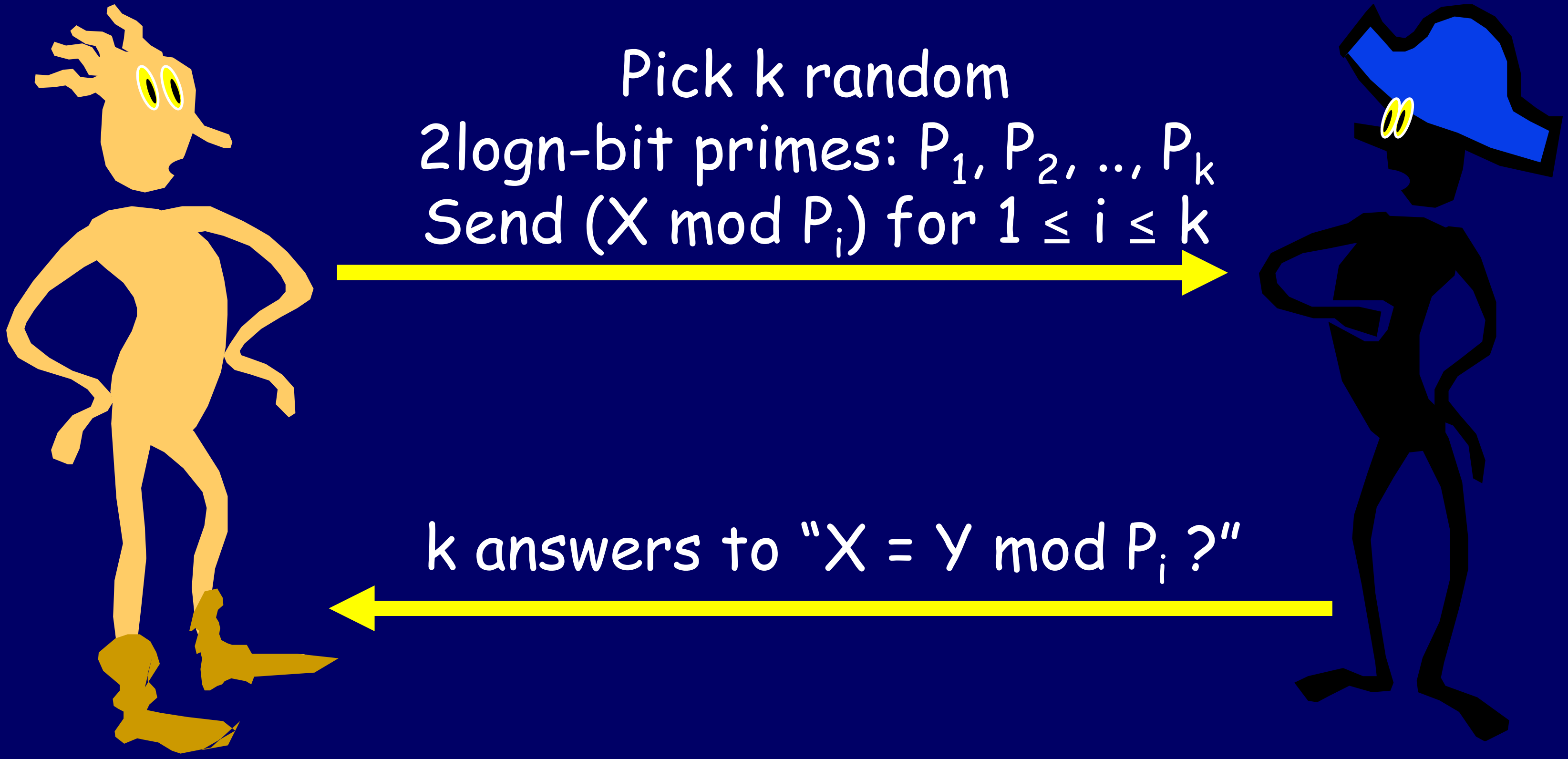
Then

$\text{Prob}[X = Y \bmod p] < 1/2$



Earth-Moon protocol makes mistake with probability at most 1/2!

Are X and Y the same n-bit numbers?



Pick k random

2logn-bit primes: $P_1, P_2, .., P_k$

Send (X mod P_i) for $1 \leq i \leq k$

k answers to “$X = Y \bmod P_i$?”

EARTH: X

MOON: Y

Exponentially smaller error probability

If X=Y, always accept.

If $X \neq Y$,

Prob $[X = Y \bmod P_i$ for all $i] \leq (1/2)^k$

Picking A Random Prime

"Pick a random n-bit prime."

Strategy:

1) Generate random n-bit numbers
2) Test each one for primality

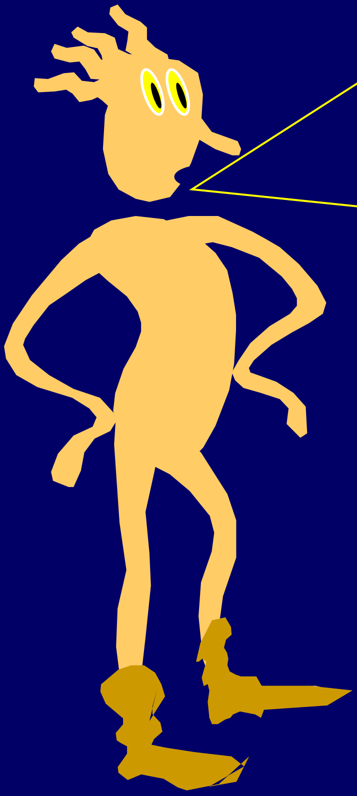
How can we test primality efficiently?

Primality Testing: Trial Division On Input n

Trial division up to $\sqrt{n}$

```
for $k = 2$ to $\sqrt{n}$ do
  if $k \mid n$ then
    return "$n$ is not prime"
  otherwise return "$n$ is prime"
```

about $\sqrt{n}$ divisions

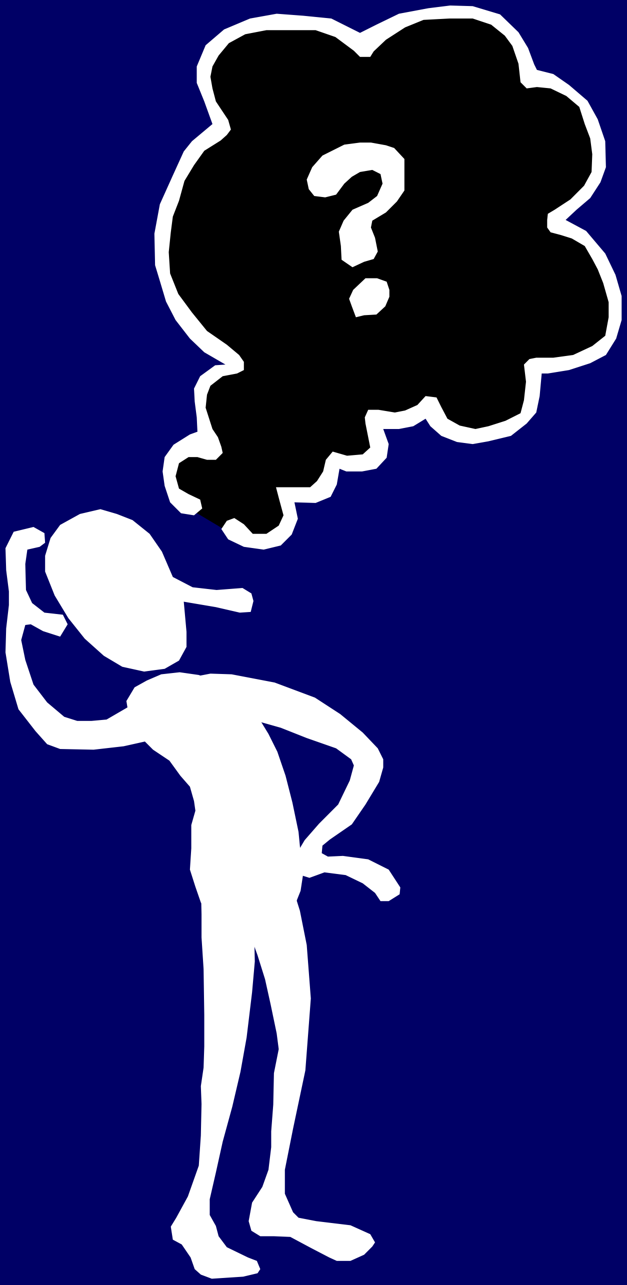


Trial division performs $\sqrt{n}$ divisions on input n.

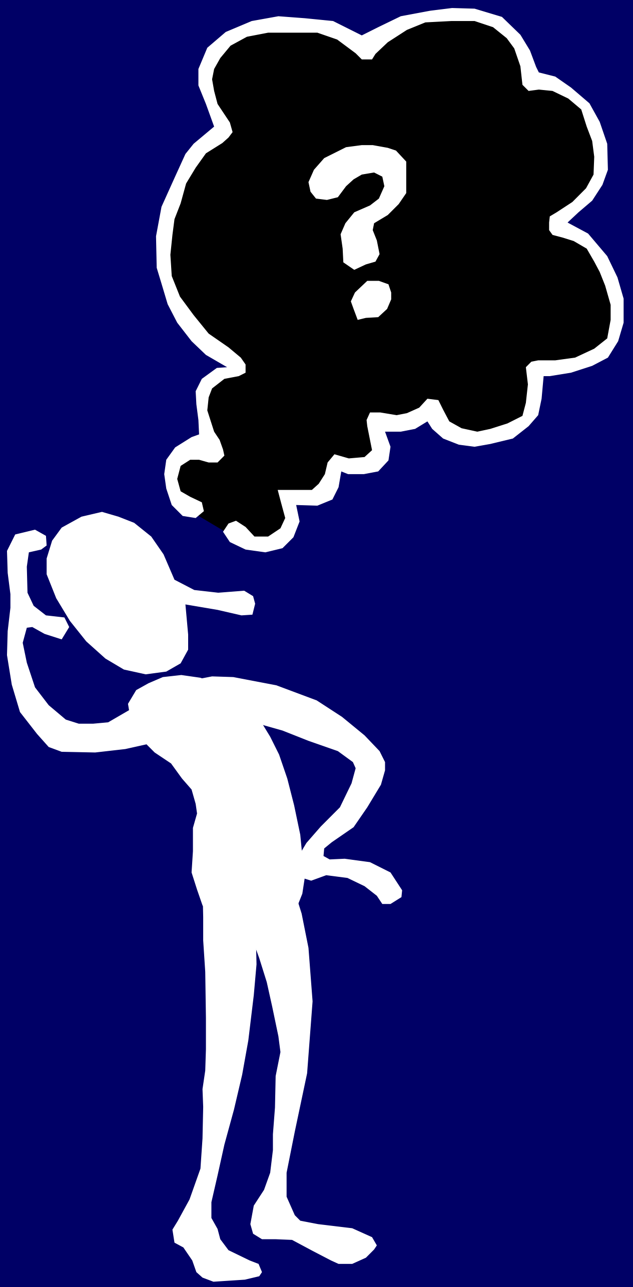
Is that efficient?

For a 1000-bit number, this will take about 2^{500} operations.

That's not very efficient at all!!!



Do the primes have a fast decision algorithm?



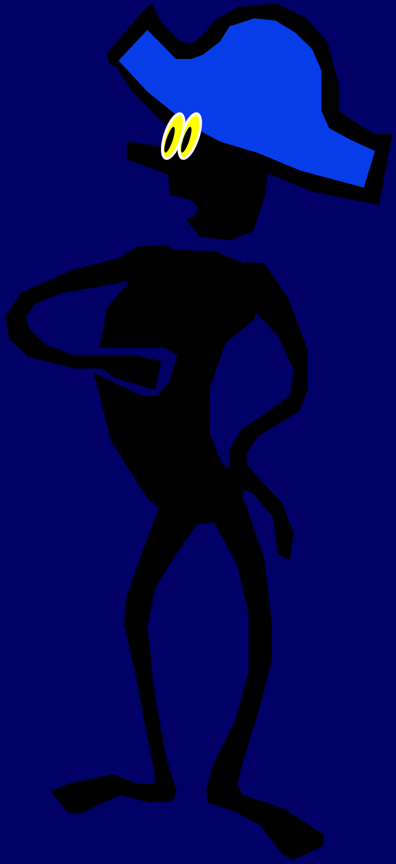
Euclid gave us a fast GCD algorithm.

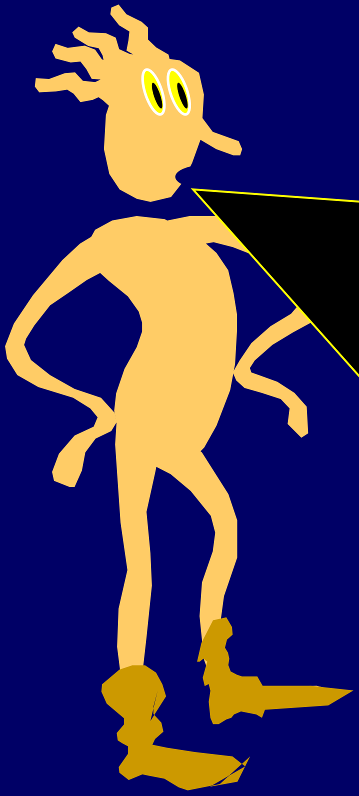
Surely, he tried to give a faster primality test than trial division.

But Euclid, Euler, and Gauss all failed!

But so many cryptosystems, like RSA and PGP, use fast primality testing as part of their subroutine to generate a random n-bit prime!

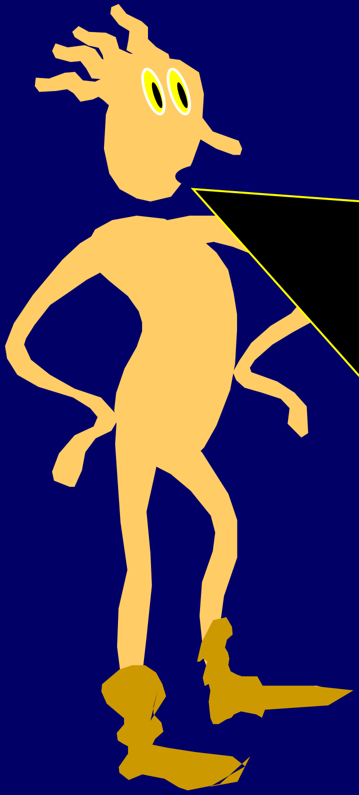
What is the fast primality testing algorithm that they use?





There are fast *randomized* algorithms to do primality testing.

Strangely, by allowing our computational model an extra instruction for flipping a fair coin, we seem to be able to compute some things faster!



If n is composite, what would be a certificate of compositeness for n?

A non-trivial factor of n.

But… even using randomness, no one knows how to find a factor quickly.

We will use a *different* certificate of compositeness that does not require factoring.

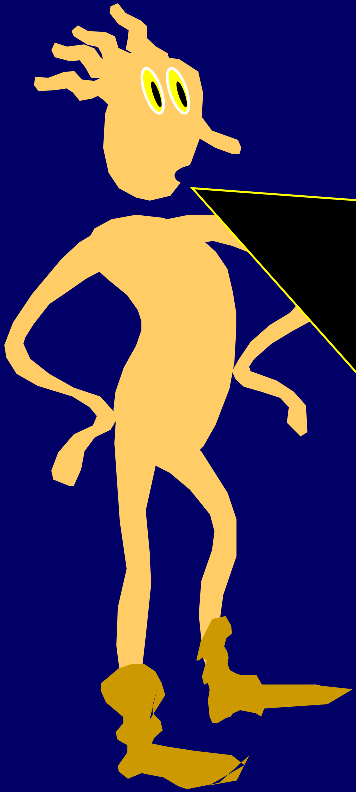
Recall that: for prime p.

Fermat: $a^{p-1} = 1 \bmod p$. $a \not\equiv 0 \pmod{p}$

When working modulo prime p, for any $a \neq 0$, $a^{(p-1)/2} =$ §1.

$X^2 = 1 \bmod p$ has at most 2 roots.

1 and -1 are roots, so it has no others.

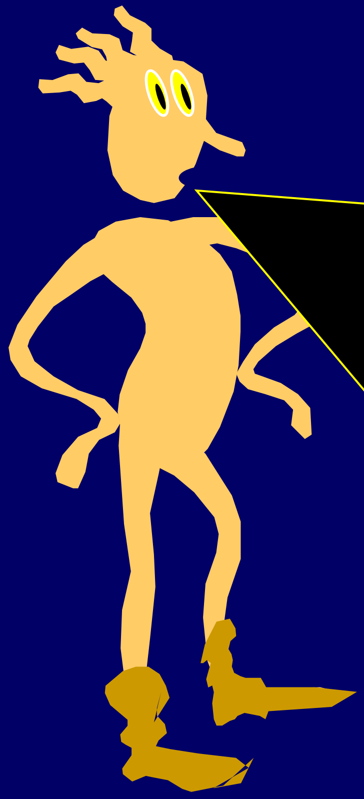


"Euler Certificate" Of Compositeness

When working modulo a prime p, for any $a \neq 0$, $a^{(p-1)/2} =$ §1.

We say that a is a certificate of compositeness for n, if $a \neq 0$ and $a^{(n-1)/2} \neq$ §1.

Clearly, if we find a certificate of compositeness for n, we know that n is composite.

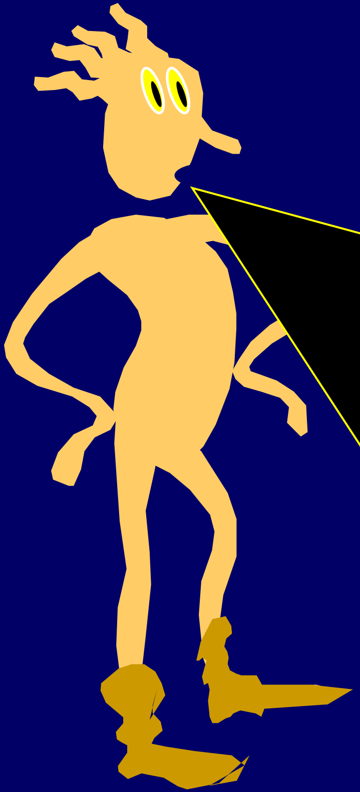


“Euler Certificates” Of Compositeness

$$EC_n = \{ a\ 2\ Z^*_n \mid a^{(n-1)/2} \neq \S 1 \}$$

certificates that n is not prime

$$\text{NOT-}EC_n = \{ a\ 2\ Z^*_n \mid a^{(n-1)/2} = \S 1 \}$$



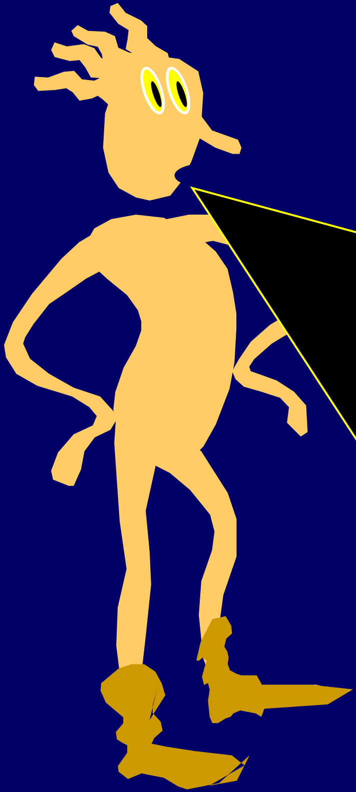
If NOT-$EC_n \neq Z^*_n$ then
EC_n is at least half of Z^*_n

In other words,
if EC_n is not empty, then
EC_n contains at least half of Z_n^*.

“Euler Certificates” Of Compositeness

$$EC_n = \{ a\ 2\ Z^*_n \mid a^{(n-1)/2} \neq \S 1 \}$$

$$NOT\text{-}EC_n = \{ a\ 2\ Z^*_n \mid a^{(n-1)/2} = \S 1 \}$$



If NOT-EC$_n \neq Z^*_n$ then
EC$_n$ is at least half of Z^*_n

In other words,
if EC$_n$ is not empty, then
EC$_n$ contains <u>at least half</u> of Z_n^*.

Randomized Primality Test

Let's suppose that EC_n contains at least half the elements of Z^*_n.

Randomized Test:

```
For i = 1 to k:
    Pick random $a_i$ 2 [2 .. n-1];
    If GCD($a_i$, n) $\neq$ 1, Halt with "Composite";
    If $a_i^{(n-1)/2} \neq$ §1 , Halt with "Composite";
```

Halt with "I think n is prime. I am only wrong $(\frac{1}{2})^k$ fraction of times I think that n is prime."

Is EC_n non-empty for all primes n?

$EC_n = \{a \in \mathbb{Z}_n^* \mid a^{n-1/2} \neq \pm 1\}$

Unfortunately, no.

Certain numbers *masquerade* as primes.

A Carmichael number is a number n such that
$a^{n-1} = 1 \pmod{n}$ for all numbers a with gcd(a,n)=1.

Example: n = 561 =3*11*17 (the smallest Carmichael number)
1105 = 5*13*17
1729 = 7*13*19

And there are many of them. For sufficiently large m, there are at least $m^{2/7}$ Carmichael numbers between 1 and m.

The saving grace

The randomized test fails only for Carmichael numbers.

But, there is an efficient way to test for Carmichael numbers.

Which gives an efficient algorithm for primality.

Randomized Primality Test

Let's suppose that EC_n contains at least half the elements of Z^*_n.

Randomized Test:

```
For i = 1 to k:
    Pick random $a_i$ 2 [2 .. n-1];
    If GCD($a_i$, n) $\neq$ 1, Halt with "Composite";
    If $a_i^{(n-1)/2} \neq$ §1 , Halt with "Composite";

    If n is Carmichael, Halt with "Composite"
Halt with "I think n is prime. I am only wrong $(\frac{1}{2})^k$ fraction
    of times I think that n is prime."
```

Randomized Algorithms

The test we outlined made one-sided error:

- It never makes an error when it thinks n is composite.
- It could just be unlucky when it thinks n is prime.

Another one-sided algorithm that never makes a mistake when it thinks n is prime.

Yet another algorithm makes 2-sided error.

- Sometimes it is mistaken when it thinks n is prime, sometimes it is mistaken when it thinks n is composite.

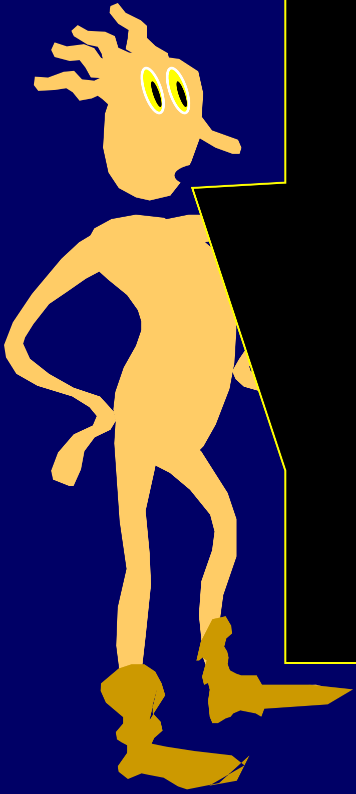
n prime means half of a's satisfy $a^{(n-1)/2} = -1 \bmod n$

If n is prime, then Z_n^* has a generator g. Then $g^{(n-1)/2} = -1 \bmod n$.

A random a2 Z_n^* is given by g^r for uniformly distributed r.

Half the time, r is odd:

$(g^r)^{(n-1)/2} = -1 \bmod n$



Another Randomized Primality Test

Suppose n is not even, nor is it the power of a number.

Randomized Test:

```
For i = 1 to k:
    Pick random $a_i$ 2 [2 .. n-1];
    If GCD($a_i$, n) ≠ 1, Halt with "Composite";
    If $a_i^{(n-1)/2}$ ≠ §1 , Halt with "Composite";

    If all k values of $a_i^{(n-1)/2}$ = +1, Halt with "I think n is composite.
        I am only wrong $(\frac{1}{2})^k$ fraction of the times."

Halt with "I think n is prime. I am only wrong $(\frac{1}{2})^k$ fraction
    of times I think that n is prime."
```

We can prove that if n is an odd composite, not a power, and there is some a such that $a^{(n-1)/2} = -1$, then $EC_n \neq ;$.

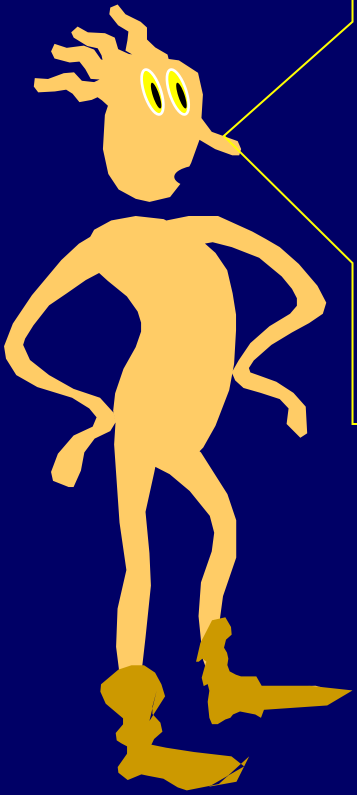
Hence, EC_n is at least a half fraction of Z^*_n.

This algorithm makes 2-sided error.
Sometimes it is mistaken when it thinks n is prime,
sometimes it is mistaken when it thinks n is composite.

Many Randomized Tests

Miller-Rabin test

Solovay-Strassen test



In 2002, Agrawal, Saxena, and Kayal (AKS) gave a deterministic primality test that runs in time $O((\text{logn})^{12})$.

This was the first *deterministic* polynomial-time algorithm that didn't depend on some *unproven conjecture*, like the Riemann Hypothesis!

Picking A Random Prime

"Pick a random n-bit prime."

Strategy:

1) Generate random n-bit numbers
2) Do fast randomized test for primality

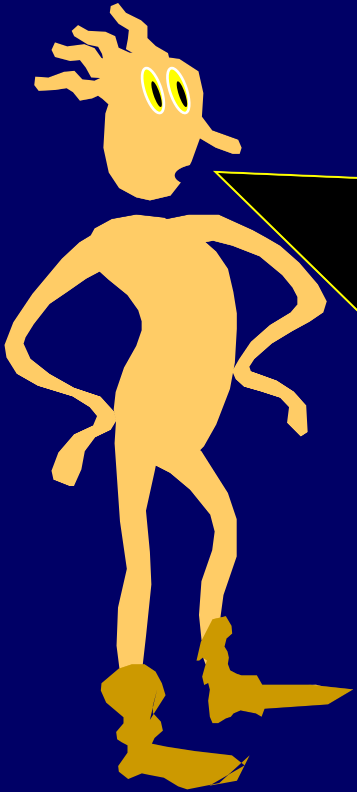
$e^{(n \log n \log\log n)^{1/3}}$

Primality Testing Versus Factoring

Primality has a fast randomized algorithm.

Factoring is not known to have a fast algorithm.

In fact, after thousands of years of research, the fastest randomized algorithm takes exp(O(n log n log n) $^{1/3}$) operations on numbers of length n. With great effort, we can currently factor 200 digit numbers.



number	digits	prize	factored
RSA-100	100		Apr. 1991
RSA-110	110		Apr. 1992
RSA-120	120		Jun. 1993
RSA-129	129	$100	Apr. 1994
RSA-130	130		Apr. 10, 1996
RSA-140	140		Feb. 2, 1999
RSA-150	150		Apr. 16, 2004
RSA-155	155		Aug. 22, 1999
RSA-160	160		Apr. 1, 2003
RSA-200	200		May 9, 2005
RSA-576	174	$10,000	Dec. 3, 2003
RSA-640	193	$20,000	Nov 2, 2005
RSA-704	212	$30,000	open
RSA-768	232	$50,000	open
RSA-896	270	$75,000	open
RSA-1024	309	$100,000	open
RSA-1536	463	$150,000	open
RSA-2048	617	$200,000	open

Google: RSA Challenge Numbers

Miller-Rabin test

The idea is to use a “converse” of Fermat’s Theorem. We know that:

$$a^{n-1} \equiv_n 1$$

for any prime n and any a in [2, n-1]. What if we try this for some number a and it fails. Then we know that n is NOT prime. Miller-Rabin is based on this idea.

Say we write n-1 as d $*2^s$ where d is odd.

Consider the following sequence of numbers mod n:

$$a^d \, , a^{2d}, a^{4d} . . . a^{d*2^{(s-1)}}, a^{d*2^S} = a^{n-1} \equiv_n 1$$

Each element is the square of the previous one.

$$a^{d}, a^{2d}, a^{4d} \ldots a^{d*2^{(s-1)}}, a^{d*2^{s}} = a^{n-1} \equiv_{n} 1$$

If n is prime, then at some point the sequence hits 1 and stays there from then on.

The interesting point is: what is the number right before the first 1. If n is prime this MUST BE n-1.

Miller-Rabin Test

To test a number n, we pick a random a and generate the above sequence. If the sequence does not hit 1, then n is composite. If there’s an element before the first 1 and it’s not n-1, then n is composite.

Otherwise n is “probably prime”.

Miller-Rabin Analysis

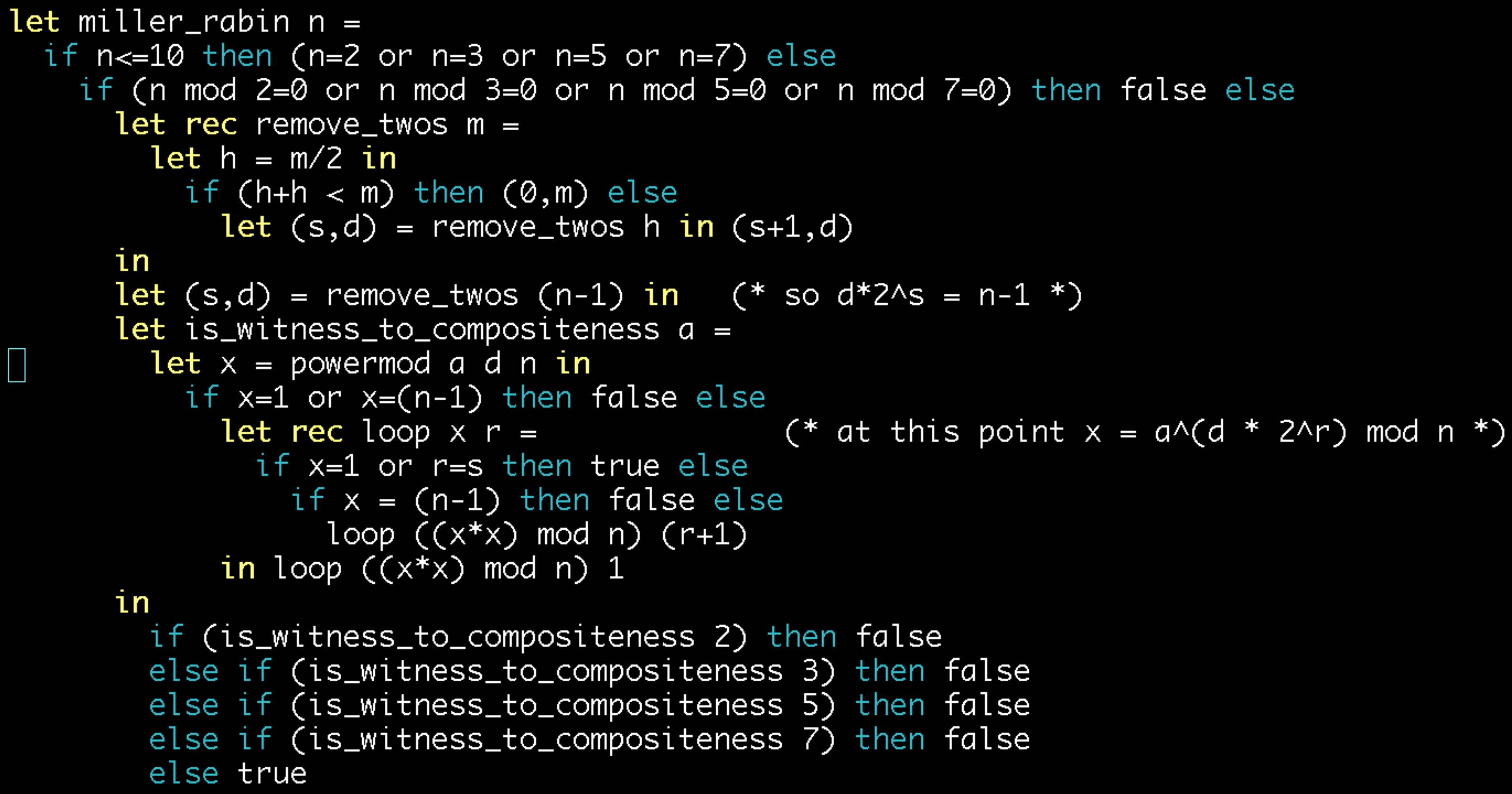
If n is composite, then with a random a, the Miller-Rabin algorithm says "composite" with probability at least 3/4 .

So if we run the test 30 times and it never says "composite" then n is prime with "probability" $1-2^{-60}$

In other words it's more likely that you'll win the lottery three days in a row than that this is giving a wrong answer.

i.e. not bloody likely.

This ocaml implementation of the Miller-Rabin test does not pick random random witnesses, but rather uses 2, 3, 5, and 7. It’s guaranteed to work up to about 2 billion. See the accompanying file big_number.ml for a full high precision implementation of Miller-Rabin with random witnesses.



```
let miller_rabin n =
  if n<=10 then (n=2 or n=3 or n=5 or n=7) else
    if (n mod 2=0 or n mod 3=0 or n mod 5=0 or n mod 7=0) then false else
      let rec remove_twos m =
        let h = m/2 in
          if (h+h < m) then (0,m) else
            let (s,d) = remove_twos h in (s+1,d)
      in
      let (s,d) = remove_twos (n-1) in   (* so d*2^s = n-1 *)
      let is_witness_to_compositeness a =
        let x = powermod a d n in
          if x=1 or x=(n-1) then false else
            let rec loop x r =              (* at this point x = a^(d * 2^r) mod n *)
              if x=1 or r=s then true else
                if x = (n-1) then false else
                  loop ((x*x) mod n) (r+1)
            in loop ((x*x) mod n) 1
      in
        if (is_witness_to_compositeness 2) then false
        else if (is_witness_to_compositeness 3) then false
        else if (is_witness_to_compositeness 5) then false
        else if (is_witness_to_compositeness 7) then false
        else true
```